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Effective Average Action of Chern-Simons Field Theory

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Abstract

The renormalization of the Chern-Simons parameter is investigated by using an exact and manifestly gauge invariant evolution equation for the scale-dependent effective average action.

1. Introduction

Pure Chern-Simons field theory in 3 dimensions is a fascinating topic from many points of view. It can be used to give a path-integral representation of knot and link invariants [1] and in order to understand many properties of 2-dimensional conformal field theories [1, 2]. Being a topological field theory the model has no propagating degrees of freedom. In fact, canonical quantization [3] yields a Hilbert space with only finitely many physical states which can be related to the conformal blocks of (rational) conformal field theories. Perturbative covariant quantization [4, 5, 6, 7, 8, 9] shows that the theory is not only renormalizable but even ultraviolet finite. It is remarkable that despite this high degree of "triviality" the theory produces nontrivial radiative corrections. Pisarski and Rao [4] and Witten [1] showed that one-loop effects lead to a renormalization of the parameter κ which multiplies the Chern-Simons 3-form in the action,

$$S_{\text{CS}}[A] = i\kappa \frac{g^2}{8\pi} \int d^3x \ \varepsilon_{\alpha\beta\gamma} [A_\alpha^a \partial_\beta A_\gamma^a + \frac{1}{3}gf^{abc} A_\alpha^a A_\beta^b A_\gamma^c] \quad (1)$$

A variety of gauge invariant regularization methods, including spectral flow arguments based upon the η -invariant, predict a finite difference between the bare and the renormalized value of κ :

$$\kappa_{\text{ren}} = \kappa_{\text{bare}} + \text{sign}(\kappa) T(G) \quad (2)$$

Here $T(G)$ denotes the value of the quadratic Casimir operator of the gauge group G in the adjoint representation. It is normalized such that $T(SU(N)) = N$. The shift of κ has a natural relation to similar shifts in the Sugawara construction of 2-dimensional conformal field theories. On the other hand, in standard renormalization theory a relation of the type (2) is rather unusual. In a generic renormalizable but not necessarily finite theory the divergent parts of the counterterms are fixed by the requirement that the renormalized quantities should be finite. Their finite parts are not fixed by any general principle but rather depend on the renormalization scheme. It was argued that, as there are no ultraviolet

divergences in Chern-Simons theory, there exists a distinguished natural renormalization scheme which leads to $\kappa_{\text{ren}} = \kappa_{\text{bare}}$ [6]. This contradicts the relation (2) favored by conformal field theory, but it is clear that any argument in favor of one of the two possibilities must come from considerations which lie outside the standard framework of renormalized perturbation theory.

In this paper we investigate Chern-Simons theory along the lines of the Wilsonian renormalization group approach by using an exact evolution equation for gauge theories which was introduced recently [10, 11]. It describes the scale dependence of the effective average action Γ_k which can be thought of as a continuous interpolation between the classical action $S \equiv \Gamma_{k \rightarrow \infty}$ and the conventional effective action $\Gamma \equiv \Gamma_{k \rightarrow 0}$. It depends on the infrared cutoff scale k in such a way that the functional Γ_k evolves out of the classical action by integrating out only those quantum fluctuations which have momenta larger than k . When k is lowered from infinity to zero, Γ_k follows a certain trajectory in the space of all actions. This trajectory is a solution of the exact renormalization group equation [10]

$$\begin{aligned} k \frac{d}{dk} \Gamma_k[A, \bar{A}] &= \frac{1}{2} \text{Tr} \left[\left(\Gamma_k^{(2)}[A, \bar{A}] + R_k(\Delta[\bar{A}]) \right)^{-1} k \frac{d}{dk} R_k(\Delta[\bar{A}]) \right] \\ &\quad - \text{Tr} \left[\left(-D_\mu[A] D_\mu[\bar{A}] + R_k(-D^2[\bar{A}]) \right)^{-1} k \frac{d}{dk} R_k(-D^2[\bar{A}]) \right] \end{aligned} \quad (3)$$

We use the background gauge fixing technique [12]. Therefore Γ_k depends on two gauge fields: the usual classical average field A_μ^a and the background field \bar{A}_μ^a . Eq.(3) has to be solved subject to the initial condition

$$\Gamma_\infty[A, \bar{A}] = S[A] + \frac{1}{2\alpha} \int d^d x \left(D_\mu^{ab}[\bar{A}] (A_\mu^a - \bar{A}_\mu^a) \right)^2 \quad (4)$$

where the classical action is augmented by the background gauge fixing term. Furthermore, $\Gamma_k^{(2)}[A, \bar{A}]$ denotes the matrix of the second functional derivatives of Γ_k with respect to A . The function R_k specifies the precise form of the infrared cutoff. It has to satisfy $\lim_{u \rightarrow 0} R_k(u) = k^2$, but is arbitrary otherwise. A convenient choice is

$$R_k(u) = u [\exp(u/k^2) - 1]^{-1} \quad (5)$$

but in some cases even a simple constant $R_k = k^2$ is sufficient. Observable quantities will not depend on the form of R_k . A similar remark applies to the precise form of the operator $\Delta[\bar{A}] \equiv -D^2[\bar{A}] + \dots$ which is essentially the covariant laplacian, possibly with additional nonminimal terms [10, 11]. The rôle of Δ is to distinguish “high momentum modes” from “low momentum modes”. If one expands all quantum fluctuations in terms of the eigenmodes of Δ , then it is the modes with eigenvalues larger than k^2 which are integrated out in Γ_k . The solution $\Gamma_k[A, \bar{A}]$ of (3) with (4) is gauge invariant under simultaneous gauge transformations of A and \bar{A} . In practice solutions can be found by truncating the space of actions to a finite dimensional subspace. If one makes an ansatz for Γ_k which contains only finitely many parameters (depending on k) and inserts it into (3), the functional differential equation reduces to a set of coupled ordinary differential equations for the parameter functions [11, 13].

The effective average action Γ_k is closely related to a continuum version of the block-spin action of lattice systems¹. Block-spin transformations can be iterated, and when we have already constructed Γ_{k_1} at a certain scale k_1 we may view Γ_{k_1} as the “classical” action for the next step of the iteration, in which an integral over $\exp(-\Gamma_{k_1})$ has to be performed. If we now apply this machinery to Chern-Simons field theory and try to understand the shift (2) from a renormalization group point of view, we are immediately confronted with the following puzzle. Because S_{CS} is not invariant under large gauge transformations, $\exp(-S_{\text{CS}})$ is single valued only if $\kappa \in \mathbf{Z}$. In the renormalization group language κ_{bare} has to be identified with $\kappa(k = \infty)$ and κ_{ren} with $\kappa(k = 0)$, where $\kappa = \kappa(k)$ is the scale-dependent prefactor of the Chern-Simons term. If there is a smooth interpolation between $\kappa(\infty)$ and $\kappa(0)$ a nontrivial shift (2) implies that there are intermediate scales at which κ cannot be integer. Hence it seems that there should be an inconsistency if we try to do the next blockspin transformation starting from a scale k_1 where κ is non-integer, because we would have to integrate over a multivalued “Boltzmann

¹Also in ref.[9] a version of the Wilsonian effective action was used.

factor” $\exp(-\Gamma_{k_1})$. Thus one is led to believe that any sensible solution to the evolution equation should have $\kappa_{\text{ren}} = \kappa_{\text{bare}}$. In the following we shall see that this “no go theorem” is actually wrong: a non-zero shift is not in contradiction with a well-defined (albeit somewhat unusual) renormalization group trajectory.

2. Truncating the Evolution Equation

Let us try to solve the initial value problem (3) with (4) for the classical Chern-Simons action (1). We work on flat euclidean space and allow for an arbitrary semi-simple, compact gauge group G . Our strategy for finding solutions of the evolution equation is to restrict the infinite dimensional space of all actions to a finite dimensional subspace by means of an appropriate ansatz for Γ_k . In the case at hand the essential physics is captured by a Γ_k of the form

$$\begin{aligned} \Gamma_k[A, N, \bar{A}] = & i\kappa(k) \frac{g^2}{4\pi} I[A] + \kappa(k) \frac{g^2}{8\pi} \int d^3x \left\{ iN^a D_\mu^{ab}[\bar{A}] (A_\mu^b - \bar{A}_\mu^b) \right. \\ & \left. - i(A_\mu^a - \bar{A}_\mu^a) D_\mu^{ab}[\bar{A}] N^b + \alpha \kappa(k) \frac{g^2}{4\pi} N^a N^a \right\} \end{aligned} \quad (6)$$

with

$$I[A] \equiv \frac{1}{2} \int d^3x \varepsilon_{\alpha\beta\gamma} [A_\alpha^a \partial_\beta A_\gamma^a + \frac{1}{3} g f^{abc} A_\alpha^a A_\beta^b A_\gamma^c] \quad (7)$$

The first term on the RHS of (6) is the Chern-Simons action, but with a scale-dependent prefactor. In the second term we introduced an auxiliary field $N^a(x)$ in order to linearize the gauge fixing term. By eliminating N^a one recovers the classical, k -independent background gauge fixing term $\frac{1}{2\alpha} (D_\mu[\bar{A}](A_\mu - \bar{A}_\mu))^2$. In principle also the gauge fixing term could change its form during the evolution, but this effect is neglected here. The ansatz (6) is motivated by the success of similar truncations in 4 dimensions. Apart from the gauge fixing term we keep only the dimension-3 operator and neglect all terms which are “irrelevant” according to their canonical dimension. It was demonstrated already that in QCD [10, 11] and in the abelian Higgs model [13] the approximation of keeping only the relevant

and the marginal terms can lead to rather accurate results which go well beyond a one-loop calculation.

For $k \rightarrow \infty$, and upon eliminating N^a , the ansatz (6) reduces to (4) with the identification $\kappa(\infty) \equiv \kappa_{\text{bare}}$. We shall insert (6) into the evolution equation and from the solution for the function $\kappa(k)$ we shall be able to determine the renormalized parameter $\kappa(0) \equiv \kappa_{\text{ren}}$. We have to project the traces on the RHS of (3) on the subspace spanned by the truncation (6). In practice this means that we have to extract only the term proportional to $I[A]$ and to compare the coefficients of $I[A]$ on both sides of the equation. In the formalism with the auxiliary field N^a , $\Gamma_k^{(2)}$ in (3) denotes the matrix of second functional derivatives with respect to both A_μ^a and N , but with \bar{A}_μ^a fixed [10]. As we are only interested in the coefficient of $I[A]$, it is computationally advantageous to set $\bar{A} = A$ after the derivatives have been performed. Then the second variation of (6) becomes

$$\begin{aligned} \delta^2 \Gamma_k[A, N, A] = & i\kappa(k) \frac{g^2}{4\pi} \int d^3x \left\{ \delta A_\mu^a \varepsilon_{\mu\nu\alpha} D_\alpha^{ab} \delta A_\nu^b + \delta N^a D_\mu^{ab} \delta A_\mu^b \right. \\ & \left. - \delta A_\mu^a D_\mu^{ab} N^b \right\} + \alpha (\kappa(k) \frac{g^2}{4\pi})^2 \int d^3x \delta N^a \delta N^a \end{aligned} \quad (8)$$

In order to facilitate the calculations we introduce three 4×4 matrices γ_μ with matrix elements $(\gamma_\mu)_{mn}$, $m=(\mu,4)=1,\dots,4$, etc., in the following way[9]:

$$\begin{aligned} (\gamma_\mu)_{\alpha\beta} &= \varepsilon_{\alpha\mu\beta}, \quad (\gamma_\mu)_{4\alpha} = -(\gamma_\mu)_{\alpha 4} = \delta_{\mu\alpha} \\ (\gamma_\mu)_{44} &= 0 \end{aligned} \quad (9)$$

If we combine the gauge field fluctuation and the auxiliary field into a 4-component object $\Psi_m^a \equiv (\delta A_\mu^a, \delta N^a)$ and choose the gauge $\alpha = 0$, then (8) assumes the form

$$\delta^2 \Gamma_k[A, N, A] = i\kappa(k) \frac{g^2}{4\pi} \int d^3x \Psi_m^a (\gamma_\mu)_{mn} D_\mu^{ab} \Psi_n^b \quad (10)$$

so that in matrix notation

$$\Gamma_k^{(2)} = i\kappa(k) \frac{g^2}{4\pi} \mathcal{D} \quad (11)$$

Clearly $\mathcal{D} \equiv \gamma_\mu D_\mu$ is reminiscent of a Dirac operator. In fact, the algebra of the γ -matrices is similar to the one of the Pauli matrices: $\gamma_\mu \gamma_\nu = -\delta_{\mu\nu} + \varepsilon_{\mu\nu\alpha} \gamma_\alpha$.

Because $\gamma_\mu^+ = -\gamma_\mu$, \not{D} is hermitian. Its square reads

$$\not{D}^2 = -D^2 - ig \, {}^*F_\mu \gamma_\mu \quad (12)$$

where ${}^*F_\mu \equiv \frac{1}{2}\varepsilon_{\mu\alpha\beta}F_{\alpha\beta}$ is the dual of the field strength tensor. (In equations such as (11) and (12) A_μ and $F_{\mu\nu}$ are matrices in the adjoint representation.) Because \not{D}^2 is ‘almost’ equal to the covariant laplacian, it is the natural candidate for the cutoff operator Δ . With this choice the evolution equation (3) reads at $\bar{A} = A$:

$$\begin{aligned} ic \, k \frac{d}{dk} \kappa(k) \, I[A] &= \frac{1}{2} \text{Tr} \left[\left(i c \kappa \not{D} + R_k(\not{D}^2) \right)^{-1} k \frac{d}{dk} R_k(\not{D}^2) \right] \\ &\quad - \text{Tr} \left[\left(-D^2 + R_k(-D^2) \right)^{-1} k \frac{d}{dk} R_k(-D^2) \right] \end{aligned} \quad (13)$$

Here $c \equiv g^2/4\pi$. The equality sign in (13) is to be understood in the sense that the term $\sim iI[A]$ has to be extracted from the RHS and all other terms have to be discarded. In particular, the second trace on the RHS of (13) is manifestly real, so it cannot match the purely imaginary $iI[A]$ and can be omitted therefore. For the same reason we may replace the first trace by i times its imaginary part:

$$k \frac{d}{dk} \kappa(k) \, I[A] = -\frac{1}{2} \kappa(k) \text{Tr} \left[\not{D} \left(c^2 \kappa^2 \not{D}^2 + R_k^2(\not{D}^2) \right)^{-1} k \frac{d}{dk} R_k(\not{D}^2) \right] + \dots \quad (14)$$

The trace in (14) involves an integration over spacetime, a summation over adjoint group indices, and a ‘Dirac trace’. We shall evaluate it explicitly in the next section. Before turning to that let us first look at the general structure of eq.(14).

In terms of the (real) eigenvalues λ of \not{D} eq.(14) reads

$$\frac{d\kappa(k)}{dk^2} \, I[A] = -\frac{1}{2} \kappa(k) \sum_\lambda \frac{\lambda}{c^2 \kappa^2(k) \lambda^2 + R_k^2(\lambda^2)} \cdot \frac{dR_k(\lambda^2)}{dk^2} \quad (15)$$

where we switched from k to k^2 as the independent variable. We observe that the sum in (15) is related to a regularized form of the spectral asymmetry of \not{D} . We emphasize at this point that the evolution equation (3), and therefore also (15), is well-defined, both in the infrared and the ultraviolet, without any further regularization. If one employs a cutoff function $R_k(u)$ which vanishes

exponentially fast for $u \rightarrow \infty$ (such as (5) for example) only eigenvalues of Δ in a small neighborhood of $\lambda \approx k$ contribute significantly to the trace [10].

An approximate solution for $\kappa(k)$ can be obtained by integrating both sides of eq.(15) from a low scale k_0^2 to a higher scale Λ^2 and approximating $\kappa(k) \simeq \kappa(k_0)$ on the RHS. (In more conventional theories [13] this type of approximation amounts to neglecting anomalous dimensions.) This yields

$$[\kappa(k_0) - \kappa(\Lambda)] I[A] = \frac{1}{2} \kappa(k_0) \sum_{\lambda} \int_{k_0^2}^{\Lambda^2} dk^2 \frac{dR_k(\lambda^2)}{dk^2} \cdot \frac{\lambda}{c^2 \kappa^2(k_0) \lambda^2 + R_k^2(\lambda^2)} \quad (16)$$

Upon using R_k as the variable of integration one arrives at

$$[\kappa(k_0) - \kappa(\Lambda)] I[A] = \frac{1}{2c} \operatorname{sign}(\kappa(k_0)) \sum_{\lambda} \operatorname{sign}(\lambda) G(\lambda; k_0, \Lambda) \quad (17)$$

with

$$G(\lambda; k_0, \Lambda) \equiv \arctan \left[c |\kappa(k_0) \lambda| \frac{R_{\Lambda}(\lambda^2) - R_{k_0}(\lambda^2)}{c^2 \kappa(k_0)^2 \lambda^2 + R_{\Lambda}(\lambda^2) R_{k_0}(\lambda^2)} \right] \quad (18)$$

Recalling the properties of R_k we see that in the spectral sum (17) the contributions of eigenvalues $|\lambda| \ll k_0$ and $|\lambda| \gg \Lambda$ are strongly suppressed, and only the eigenvalues with $k_0 < |\lambda| < \Lambda$ contribute effectively. Ultimately we would like to perform the limits $k_0 \rightarrow 0$ and $\Lambda \rightarrow \infty$. In this case the sum over λ remains without IR and UV regularization. This means that if we want to formally perform the limits $k_0 \rightarrow 0$ and $\Lambda \rightarrow \infty$ in eq.(17), we have to introduce an alternative regulator. In order to make contact with the standard spectral flow argument [1] let us briefly describe this procedure. We avoid IR divergences by putting the system in a finite volume and imposing boundary conditions such that there are no zero modes. In the UV we regularize with a zeta-function-type convergence factor $|\lambda/\mu|^{-s}$ where μ is an arbitrary mass parameter. Thus the spectral sum becomes

$$\lim_{s \rightarrow 0} \sum_{\lambda} \operatorname{sign}(\lambda) \left| \frac{\lambda}{\mu} \right|^{-s} G(\lambda; k_0, \Lambda) \quad (19)$$

Now we interchange the limits $k_0 \rightarrow 0$, $\Lambda \rightarrow \infty$ and $s \rightarrow 0$. By construction, only finite ($|\lambda| \leq \mu$) and nonzero eigenvalues contribute in (19). For such λ 's we have

$G(\lambda; 0, \infty) = \pi/2$ irrespective of the precise form of R_k . Therefore (17) becomes

$$[\kappa(0) - \kappa(\infty)] I[A] = \frac{2\pi^2}{g^2} \operatorname{sign}(\kappa(0)) \eta[A] \quad (20)$$

where $\eta[A] \equiv \lim_{s \rightarrow 0} \frac{1}{2} \sum_{\lambda} \operatorname{sign}(\lambda) |\lambda/\mu|^{-s}$ is the eta-invariant. If we insert the known result [1] $\eta[A] = (g^2/2\pi^2) T(G) I[A]$ we find that in agreement with eq.(2)

$$\kappa(0) = \kappa(\infty) + \operatorname{sign}(\kappa(0)) T(G) \quad (21)$$

We see that at least at the formal level the function R_k has dropped out of the calculation. In this sense the shift of the parameter κ is universal: it does not depend on the form of the IR cutoff.

3. Explicit Calculation

Next we turn to the evaluation of the trace in eq.(14). The derivation in this section does not rely on formal manipulations of spectral sums, and it will keep the full k -dependence of κ on the RHS. It is precisely this $\kappa(k)$ -dependence on the RHS of the evolution equation which implements the “renormalization group improvement” [10, 11]. To start with we use the constant cutoff $R_k = k^2$ for which eq.(14) assumes the form²

$$\frac{d}{dk^2} \kappa(k) I[A] = -\frac{1}{2c^2 \kappa(k)} \operatorname{Tr} \left[\not{D} \left(\not{D}^2 + l(k)^2 \right)^{-1} \right] \quad (22)$$

where

$$l(k) \equiv \frac{k^2}{c |\kappa(k)|} \quad (23)$$

(Note that in 3 dimensions $c \equiv g^2/4\pi$ and hence also l has the dimension of a mass.) Our strategy is to extract from the trace the term quadratic in A and linear in the external momentum, and to equate the coefficients of the $A \partial A$ -terms on both sides. (Using the A^3 -term instead leads to the same answer.) Using

²Even with $R_k = k^2$ there are no convergence problems for $\lambda \rightarrow \infty$ in eq.(15). The extraction of the term $\sim I[A]$ from the spectral sum involves derivatives which improve the convergence, see eq.(25) below.

$tr(\gamma_\alpha \gamma_\mu \gamma_\nu) = -4\varepsilon_{\alpha\mu\nu}$, $f^{acd} f^{bcd} = T(G) \delta^{ab}$ and similar identities one obtains after some algebra

$$\frac{d\kappa(k)}{dk^2} \int d^3x \varepsilon_{\alpha\beta\gamma} A_\alpha^a \partial_\beta A_\gamma^a = -\frac{g^2 T(G)}{c^2 \kappa(k)} \int d^3x \varepsilon_{\alpha\beta\gamma} A_\alpha^a \Pi_k(-\partial^2) \partial_\beta A_\gamma^a + O(A^3) \quad (24)$$

The function Π_k is given by the Feynman parameter integral

$$\Pi_k(q^2) = 8 \int_0^1 dx \ x(1-x) \int \frac{d^3p}{(2\pi)^3} \frac{q^2}{[p^2 + l^2 + x(1-x)q^2]^3} \quad (25)$$

Expanding $\Pi_k(-\partial^2) = \Pi_k(0) - \Pi'_k(0)\partial^2 + \dots$, we see that only for the term with $\Pi_k(0)$ the number of derivatives on both sides of eq.(24) coincides. Therefore one concludes that

$$\frac{d\kappa(k)}{dk^2} = -\frac{g^2 T(G)}{c^2 \kappa(k)} \Pi_k(0) \quad (26)$$

where $\Pi_k(0)$ depends on $\kappa(k)$ via (23). Equation (26) is the renormalization group equation for $\kappa(k)$ which we wanted to derive. Formally it is similar to the evolution equations which we derived for QCD[10] and for the abelian Higgs model [13]. The very special features of Chern-Simons theory, reflecting its topological character, become obvious when we give a closer look to the function $\Pi_k(q^2)$. Assume we fix a non-zero value of k ($l \neq 0$) and let $q^2 \rightarrow 0$ in (25). Because the l^2 -term prevents the p -integral from becoming IR divergent, we may set $q^2 = 0$ in the denominator, and we conclude that the integral vanishes $\sim q^2$. This means that the RHS of (26) is zero and that $\kappa(k)$ keeps the same value for all strictly positive values of k . One might be tempted to take this result as a confirmation of the “no-go theorem” mentioned in the introduction and to conclude that $\kappa_{\text{ren}} = \kappa_{\text{bare}}$. This is premature however because $\Pi_k(0)$ really vanishes only for $k > 0$. If we set $l = 0$ in (25) we cannot conclude anymore that $\Pi_k \sim q^2$, because in the region $p^2 \rightarrow 0$ the term $x(1-x)q^2$ provides the only IR cutoff and may not be set to zero in a naive way. In fact, $\Pi_k(0)$ has a δ -function-like peak at $k = 0$. To see this, we first perform the integrals in (25):

$$\Pi_k(q^2) = \frac{1}{\pi} \left[\frac{1}{2|q|} \arctan \left(\frac{|q|}{2|l|} \right) - \frac{|l|}{q^2 + 4l^2} \right] \quad (27)$$

As q^2 approaches zero, this function develops an increasingly sharp maximum at $l = 0$. Integrating (27) against a smooth test function $\Phi(l)$ it is easy to verify that

$$\lim_{q^2 \rightarrow 0} \int_0^\infty dl \Phi(l) \Pi_k(q^2) = \frac{1}{4\pi} \Phi(0) \quad (28)$$

This means that on the space of even test functions $\lim_{q^2 \rightarrow 0} \Pi_k(q^2) = \delta(l)/2\pi$. Even though the value of $\kappa(k)$ does not change during almost the whole evolution from $k = \infty$ down to very small scales, it performs a finite jump in the very last moment of the evolution, just before reaching $k = 0$. This jump can be calculated in a well-defined manner by integrating (26) from $k^2 = 0$ to $k^2 = \infty$:

$$\kappa(0) - \kappa(\infty) = 4\pi T(G) \lim_{q^2 \rightarrow 0} \int_0^\infty dl \operatorname{sign}(\kappa(l)) \cdot \left[1 - c l \frac{d}{dk^2} |\kappa(k)| \right]^{-1} \Pi_k(q^2) \quad (29)$$

The term $\sim d|\kappa|/dk^2$ is a Jacobian factor which is due to the fact that l depends on $\kappa(k)$. This factor is the only remnant of the $\kappa(k)$ -dependence of the RHS of the evolution equation. We mentioned already that, in more conventional theories, this dependence of the RHS on the running couplings is the origin of the renormalization group improvement. Chern-Simons theory is special also in this respect. If we use (28) in (29), $l d|\kappa|/dk^2$ is set to zero and we find

$$\kappa(0) = \kappa(\infty) + \operatorname{sign}(\kappa(0)) T(G), \quad (30)$$

which is precisely the 1-loop result. It is straightforward to check that the shift (30) is independent of the choice for R_k . For a generic cutoff the momentum space integral (25) becomes more complicated and depends on R_k nontrivially. Nevertheless, by an argument similar to the one following eq.(16) the relation (30) can be seen to hold for any R_k .

4. Conclusion

We used an exact and manifestly gauge invariant evolution equation in order to study the renormalization of the Chern-Simons parameter. The method of truncating the space of actions allows us to obtain nonperturbative solutions which

require neither an expansion in the number of loops nor in the gauge coupling. The approximation involved here is that during the evolution the mixing of the Chern-Simons term with other operators is neglected. This approach has been tested already in the abelian Higgs model [13] and in QCD[10, 11]. The results obtained for Chern-Simons theory are strikingly different in at least two respects.

Like κ , also the gauge coupling in QCD_4 , for instance, is a universal quantity. Its running is governed by a R_k -independent β -function which leads to a logarithmic dependence on the scale k . The Chern-Simons parameter κ , on the other hand, does not run at all between $k = \infty$ and any infinitesimally small value of k . Only at the very end of the evolution, when k is very close to zero, κ jumps by a universal, unambiguously calculable amount $\pm T(G)$. Though surprising in comparison with non-topological theories, this feature is precisely what one would expect if one recalls the topological origin of a non-vanishing η -invariant [1]. If $\eta[A] \neq 0$ for a fixed gauge field A , some of the low lying eigenvalues of $\mathcal{D}[A]$ must have crossed zero during the interpolation from $A = 0$ to A . However, this spectral flow involves only that part of the spectrum which, in the infinite volume limit, is infinitesimally close to zero. It is gratifying to see that even without an artificial discretization of the spectrum (by a finite volume) the spectral flow is correctly described by the evolution equation. A jump in κ , rather than a continuous evolution, resolves the puzzle mentioned in the introduction: at $k > 0$ the iterated block-spin transformations are all well-defined, but their limit is nontrivial. It is also remarkable that the evolution equation by itself is well-defined even for noninteger κ . The quantization condition follows only if we require the limit $\lim_{k \rightarrow 0} \exp(-\Gamma_k)$ to be a single-valued functional³.

The second unusual feature of Chern-Simons theory is the absence of any renormalization group improvement beyond the 1-loop result. This situation has to be contrasted with the running of g in QCD_4 , for instance where a truncation

³A similar phenomenon occurs in stochastic quantization [14].

similar to the one used here leads to a nonperturbative β -function involving arbitrarily high powers of g . We emphasize that our exact evolution equation with the truncation (6) potentially goes far beyond a 1-loop calculation. It is quite remarkable therefore that in Chern-Simons theory all higher contributions vanish. It is not possible to translate such a “nonrenormalization theorem” for a given truncation into a statement about the nonrenormalization at a given number of loops. Nevertheless, our results point in the same direction as ref. [6] where the absence of 2-loop corrections was proven. As there are gauge-invariant regularizations which do not produce the shift (2) [15] it remains an open question whether more complicated truncations could modify the above picture.

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